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Anomalous couplings in M-theory and string theory

James Sparks

Department of Physics, Queen Mary, University of London
Mile End Road, London E1 4NS, UK
E-mail: J.Sparks@qmul.ac.uk

Abstract: We explore how various anomaly-cancelling terms in M-theory and string theory transform non-trivially into each other under duality. Specifically, we study the phenomenon in which bulk terms in M-theory get mapped to brane worldvolume terms in string theory. The key mathematical ingredient is G-index theory.

Keywords: M-Theory, String Duality, Anomalies in Field and String Theories.

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Contents

1. Introduction and summary

In this note we investigate how various anomalous couplings in M-theory and string theory are related to each other under duality. In particular, we focus on the phenomenon in which bulk terms in M-theory get mapped to brane worldvolume couplings in string theory.

In order to explain this, consider some configuration of D6-branes in type-IIA string theory. The D6-brane is a Kaluza-Klein monopole and so naively this configuration will lift to some geometric background in M-theory. In the effective worldvolume theory of the D6-branes there are Wess-Zumino terms which couple the brane to the bulk Ramond-Ramond fields. These couplings are related to anomaly cancellation for intersecting branes, and, in certain circumstances, can lead to induced topological charges. A natural question to ask is, therefore, what happens to these couplings when we lift to M-theory?

Alternatively, we may pose the problem the other way around. As an illustration, consider the gravitational Chern-Simons term in M-theory

$$\int C \wedge I_8(\mathcal{R})\,. \tag{1.1}$$

Here C denotes the three-form potential and $I_8(\mathcal{R})$ is a certain eight-form constructed out of the Riemann tensor $\mathcal{R}$. The term (1.1) participates in an anomaly inflow mechanism which cancels an anomaly in the M5-brane worldvolume theory [1, 2]. Reducing to type-IIA string theory on a circle, (1.1) trivially becomes [1] the one-loop interaction found in [3] on replacing C with the NS-NS two-form B which couples to the fundamental string. This interaction was originally calculated by compactifying the IIA string on an eight-dimensional manifold X and then computing the one-point function of the B-field [3].

However, we shall argue here that there is a non-trivial reduction of the anomalous term (1.1) to type-IIA which is not a Kaluza-Klein reduction in the usual sense. To explain this, recall [4] that on the worldvolume W of a D6-brane one has a coupling

$$\int_W C_3 \wedge \frac{1}{48}\left(-p_1(TW) + p_1(NW)\right) \tag{1.2}$$

where C_3 denotes the (pull-back of the) Ramond-Ramond (RR) three-form potential, TW denotes the tangent bundle of the brane worldvolume W, NW denotes the normal bundle of W in spacetime, and p_1 is the first Pontryagin class. We will argue that the terms (1.1) and (1.2) are in fact dual to each other. This is clearly not a standard Kaluza-Klein reduction, in the sense that the CI_8 coupling in M-theory reduces to BI_8 in string theory, for example.

One may argue that (1.1), (1.2) are related by a simple extension of some arguments that may be found in [3, 5]. Start in type-IIA and consider the interaction BI_8. If one "compactifies" the IIA string to M_6 on the ALE singularity A_{N-1}, one finds that this one-loop term gives rise to a quantum correction [5]

$$-\frac{N}{48}\int_{M_6} B\wedge p_1\,. \tag{1.3}$$

One may now perform a "9-11 flip", trivially lifting to M-theory on a circle $\mathbf{S}^1$, and coming down to type-IIA on an $\mathbf{S}^1$ contained in the ALE space. One ends up with N D6-branes with worldvolume $M_6\times\mathbf{S}^1$ and a brane coupling

$$-\frac{N}{48}\int_{M_6\times\mathbf{S}^1} C_3\wedge p_1\,. \tag{1.4}$$

This is precisely the gravitational Wess-Zumino coupling (1.2). Unfortunately, there is no contribution from the normal bundle in this case, since the latter is trivial. Moreover, the computation is entirely local - that is, we have only considered a neighbourhood of the D6-brane.

In this note we prove equality of the couplings (1.1) and (1.2) in a more general setting. As in [3], we consider the case of compactification on an arbitrary eight-dimensional spin manifold X. Equality of the gravitational terms (1.1), (1.2) then follows using G-index theory on X. A G-index theorem is just like a usual index theorem for an elliptic operator, but with an additional group action by G, and may be regarded as a generalisation of the Lefschetz fixed point formula. We will take $G\cong \mathrm{U}(1)$, which rotates the M-theory circle fibres. The beauty of this approach is that we prove equality of the gravitational terms in general (in the context of compactification on X), corresponding to an arbitrary configuration of D6-branes in type-IIA.

Similarly, we argue using anomaly cancellation that the Chern-Simons interaction of eleven-dimensional supergravity maps to the gauge Wess-Zumino terms for a single D6-brane. Thus, summarising (in conventions described in the next section), we therefore have what might be described as "non-standard" Kaluza-Klein reductions:

$$\begin{aligned}\int_{M_{11}} C\wedge I_8(\mathcal{R}) &\sim \int_W C_3\wedge\frac{1}{48}\left(-p_1(TW)+p_1(NW)\right)\\ \frac{1}{24\pi^2}\int_{M_{11}} C\wedge G\wedge G &\sim \int_W C_3\wedge \mathrm{ch}(E)\end{aligned} \tag{1.5}$$

from M-theory to type-IIA string theory with D6-branes, where $\mathrm{ch}(E)$ denotes the Chern character of the gauge bundle E. Moreover, by applying T-duality one recovers the gravitational and gauge couplings for general Dp-branes in type-II string theory.

The plan of the paper is as follows. In section 2 we consider compactification of M-theory on X, focusing in particular on the membrane tadpole, and the dual interpretation in type-IIA string theory. In section 3 we work out the relevant G-index theorems and use the results to prove the relations above.

2. Effective membrane charges

Our starting point is the low-energy effective action for the bosonic fields of M-theory. This takes the form

$$S = \frac{1}{2\kappa_{11}^2} \int \left(d^{11}x\sqrt{-g}R - \frac{1}{2}G \wedge *G - \frac{1}{6}C \wedge G \wedge G \right) - \int C \wedge I_8(\mathcal{R}) \tag{2.1}$$

where κ_{11}^2 is related to the eleven-dimensional Planck length by $\kappa_{11}^2 \sim l_P^9$. For a careful treatment of the numerical factors see, for example, [6]. We shall set the M2-brane tension T_2 to unity. It then follows that the M5-brane tension is given by $T_5 = 1/2\pi$ and $\kappa_{11}^2 = 2\pi^2$.

The last term in (2.1) is a gravitational Chern-Simons correction to the eleven-dimensional supergravity action. Explicitly, we have

$$\int C \wedge I_8(\mathcal{R}) = \frac{1}{192} \int C \wedge \left(p_1^2 - 4p_2\right). \tag{2.2}$$

The Pontryagin forms are given in terms of the curvature two-form $\mathcal{R}$ by

$$p_1 = -\frac{1}{8\pi^2}\mathrm{tr}\mathcal{R}^2, \quad p_2 = -\frac{1}{64\pi^4}\mathrm{tr}\mathcal{R}^4 + \frac{1}{128\pi^4}(\mathrm{tr}\mathcal{R}^2)^2. \tag{2.3}$$

In order to "see" the gravitational Chern-Simons term, as in [3] we consider compactification of M-theory on a closed eight-dimensional spin manifold X. As pointed out in [7], eight dimensions is particularly interesting as far as anomalies are concerned, which is partly why we focus attention on this case. Thus the M-theory background is of the form $M_3 \times X$ where M_3 is a three-manifold. One might typically take $M_3 = \mathbb{R}^{2,1}$, for example. Variation of the C-field gives rise to the equation of motion

$$\frac{1}{4\pi^2} d * G + \sum_{i=1}^{N_{M2}} \delta(x_i) + \frac{1}{192}\left(p_1^2(X) - 4p_2(X)\right) + \frac{1}{2}\left(\frac{G}{2\pi}\right)^2 = 0 \tag{2.4}$$

where we have included N_{M2} space-filling M2-branes in the vacuum. The i^{th} M2-brane is located at the point $x_i \in X$, and $\delta(x_i)$ denotes the Poincaré dual to the worldvolume.[1]

Integrating (2.4) over X gives rise to the anomaly cancellation condition

$$N_{M2} + \frac{1}{192}\int_X \left(p_1^2 - 4p_2\right) + \frac{1}{2}\int_X \left(\frac{G}{2\pi}\right)^2 = 0. \tag{2.5}$$

The fact that, given an arbitrary spin manifold X together with a correctly quantised G-flux, one can always solve (2.5) with N_{M2} integer is actually rather non-trivial [8].

[1] For an anti-M2-brane one has a minus sign.

JHEP07(2003)042

Suppose now that there exists a reduction of M-theory on X to type IIA string theory on Y, with D6-branes in the vacuum. This means that X admits a semi-free circle action[2] with a codimension four fixed point set B. Notice that B need not be connected. X is a circle bundle over Y, with the circle fibres degenerating over the locus B. This circle bundle is usually called the "M-theory circle bundle".

More precisely, we may construct the type-IIA manifold Y as follows. We start by deleting from X a small tubular neighbourhood surrounding B. The boundary of that tubular neighbourhood is a seven-manifold P which is a three-sphere bundle over B. When we reduce to type-IIA by quotienting out the M-theory circle, the boundary P maps to a boundary six-manifold $S = P/\,\mathrm{U}(1)$ which is a two-sphere bundle over B. We complete the IIA spacetime Y by filling in each two-sphere fibre with a three-disc, thus obtaining a closed seven-manifold.

M-theory on X is then dual to type-IIA string theory on Y with a space-filling D6-brane wrapped over the codimension three submanifold $B \subset Y$, where B is embedded as the zero-section of the disc bundle described in the last paragraph. Thus if $\mathbf{S}^2$ denotes a small two-sphere linking a connected component of B in Y we have

$$\int_{\mathbf{S}^2} \frac{G_2}{2\pi} = 1 \tag{2.6}$$

where G_2 denotes the RR two-form field strength. After reduction to type-IIA theory the effective membrane charges become the effective charge of space-filling D2-branes. Thus there must be a type-IIA interpretation of the anomaly formula (2.5).

We assume throughout that the NS-NS H-field is cohomologically trivial. This means that the integral of the M-theory G-flux over the M-theory circle is zero in $H^3(Y)$. There is therefore no contribution to the D2-brane charge from the bulk. However, in type-IIA theory we also have a space-filling D6-brane wrapped over the four-cycle $B \subset Y$. Due to the non-trivial embedding of the D6-brane worldvolume in spacetime, the Ramond-Ramond fields in the bulk couple to various terms on the D6-brane worldvolume [4]

$$I_{\mathrm{WZ}} = \int_W C_* \wedge \mathrm{ch}(E) \wedge \sqrt{\frac{\hat{A}(TW)}{\hat{A}(NW)}} \tag{2.7}$$

where $\hat{A}$ denotes the Dirac genus. Comparing the C_3 coupling in (2.7) with the formula (2.5), we see that the type IIA analogue of the latter is

$$N_{D2} + \int_B \sqrt{\hat{A}(TB)/\hat{A}(NB)} + \frac{1}{2}\int_B \left(\frac{\mathcal{F}}{2\pi}\right)^2 = 0 \tag{2.8}$$

where N_{D2} is the number of space-filling D2-branes, and the U(1) gauge field strength on the D6-brane is denoted $\mathcal{F}$. N_{D2} is naturally identified with N_{M2} in M-theory. The aim of this paper is to justify that the geometric (and flux) terms are also naturally identified. That is

$$\frac{1}{192}\int_X \left(p_1^2(X) - 4p_2(X)\right) = \frac{1}{48}\int_B \left(-p_1(TB) + p_1(NB)\right) \tag{2.9}$$

[2]That is, freely acting outside the set of points fixed under the entire group action.

and

$$\frac{1}{2}\int_X \left(\frac{G}{2\pi}\right)^2 = \frac{1}{2}\int_B \left(\frac{\mathcal{F}}{2\pi}\right)^2 . \tag{2.10}$$

The relation (2.9) between the Pontryagin numbers of X and the Pontryagin numbers of B will be proved in the next section using a combination of G-index theorems. The relation (2.10) between the G-flux in M-theory and the $\mathcal{F}$-flux on the D6-branes then follows from anomaly cancellation in M-theory and string theory. That is, after identifying N_{M2} with N_{D2} and having proven (2.9), the relation (2.10) then follows from anomaly cancellation. In fact, one can show quite generally that the gauge field on a D6-brane is determined in terms of the RR four-form by the formula

$$\frac{\mathcal{F}}{2\pi} = \int_{\mathbf{S}^2} \frac{G_4}{2\pi} \tag{2.11}$$

where $\mathbf{S}^2$ is as in (2.6), and that the Freed-Witten quantisation law for $\mathcal{F}$ then follows from subtle factors in the quantisation law for G_4. Details may be found in [9].

3. Gravitational Chern-Simons terms and G-index theorems

Our aim in this section is to prove the formula (2.9) relating the gravitational Chern-Simons term in M-theory with the gravitational Wess-Zumino terms on a D6-brane in type-IIA string theory. As we shall see, this is a consequence of a combination of G-index theorems.

The relation between index theory and anomalies is of course well-known, but G-index theory is perhaps less familiar to physicists.[3] We shall show here that anomalous terms in M-theory and anomalous couplings in string theory may be related via G-index theorems, where $G \cong \mathrm{U}(1)$ acts by rotating around the M-theory circle direction.

It is perhaps best to start with a simple example of what we mean by a G-index theorem. We take $G \cong \mathrm{U}(1)$, since this will be the case of interest. Suppose then that we have a circle action on an oriented even-dimensional manifold X, with fixed point set F. The latter need not be connected. Each connected component will have even codimension, although the dimensions of different components may differ. Then the Euler number of X may be written as the sum of the Euler numbers of each connected component F_i of F

$$\chi(X) = \sum_i \chi(F_i) \tag{3.1}$$

where the index i runs over each connected component. This is an example of a G-index theorem. The elliptic operator is just the exterior derivative. As a simple example of this particular theorem, take $X = \mathbf{S}^2$ with circle action $\mathrm{U}(1) \subset \mathrm{SO}(3)$. This rotates the two-sphere, leaving fixed the north and south poles. The Euler number of a point is 1, and so we get the well-known result that

$$\chi(\mathbf{S}^2) = \chi(\text{north pole}) + \chi(\text{south pole}) = 1 + 1 = 2\,. \tag{3.2}$$

[3] There are some notable exceptions in the literature. For example, a $\mathbb{Z}_2$-index theorem was used in [10] to compute the anomaly for a self-dual tensor.

JHEP07(2003)042

Since the group U(1) is connected, the general G-index theorem of [11] may be used to relate the index of an elliptic operator to various characteristic classes evaluated on the fixed point set. A codimension four fixed point set B then gets interpreted as a D6-brane worldvolume in type-IIA string theory. We now show how this works in detail for two elliptic operators on the closed spin eight-manifold X.

3.1 The G-signature theorem

Recall that the signature of a compact oriented eight-manifold X is defined to be the signature of the quadratic form defined on $H^4(X;\mathbb{Z})$ given by the cup product

$$H^4(X;\mathbb{Z}) \ni u \mapsto \int_X u \cup u\,. \tag{3.3}$$

Thus it is the number of (linearly independent) self-dual harmonic four-forms minus the number of (linearly independent) anti-self-dual harmonic four-forms. We denote the signature as $\mathrm{Sign}(X)$. This may be expressed as the index of an elliptic operator on X [11]. There is also a G-index theorem for this operator where we have in addition an action by the group G. The form of this G-index theorem was worked out in the original paper [11]. In order to state it, we will first of all need to establish some basic notations.

The action of U(1) on NB, the normal bundle of the fixed point set B in X, induces a natural complex structure on NB. Indeed, NB splits into a sum of two complex line bundles, and we take the U(1) action on each to be multiplication by $e^{i\theta}$, where $\theta \in [0, 2\pi]$ is the U(1) group parameter. Thus NB, with its U(1) action, is naturally a rank two complex vector bundle over B. We shall denote this bundle as $V = NB$. Thus V has Chern classes. These will be denoted by $c_1 = c_1(V) \in H^2(B;\mathbb{Z})$ and $c_2 = c_2(V) \in H^4(B;\mathbb{Z})$.

Then the G-signature theorem of [11] in this case reduces to

$$\mathrm{Sign}(X) = -\cot^2(\theta/2) \int_B L(TB) \wedge \mathcal{M}^\theta\,. \tag{3.4}$$

Here L is the usual Hirzebruch L-polynomial. It may be written in terms of Pontryagin classes as[4]

$$L = 1 + \frac{1}{3}p_1 + \frac{1}{45}\left(-p_1^2 + 7p_2\right) + \cdots\,. \tag{3.5}$$

The stable characteristic class $\mathcal{M}^\theta = \mathcal{M}^\theta(c_k)$ is a polynomial in the Chern classes of V. The coefficients are functions of θ. Specifically, we calculate[5]

$$\mathcal{M}^\theta = 1 + 2i\mathrm{cosec}(\theta)c_1 - \mathrm{cosec}^2\left(\frac{\theta}{2}\right)(c_1^2 - 2c_2) - 4\mathrm{cosec}^2(\theta)c_2 + \cdots\,. \tag{3.6}$$

The G-signature theorem thus gives

$$\mathrm{Sign}(X) = -\cot^2\left(\frac{\theta}{2}\right)\cdot \mathrm{Sign}(B) + \mathrm{cosec}^2\left(\frac{\theta}{2}\right)\cot^2\left(\frac{\theta}{2}\right)\int_B (c_1^2 - 2c_2) + \mathrm{cosec}^4\left(\frac{\theta}{2}\right)\int_B c_2 \tag{3.7}$$

[4]This is the usual definition. The definition in [11] differs by factors of 2. Our definition of $\mathcal{M}^\theta$ also therefore differs by factors of 2.

[5]The c_1 coefficient was computed in [11].

where we have used the ordinary signature theorem to write

$$\mathrm{Sign}(B) = \frac{1}{3}\int_B p_1(TB)\,. \tag{3.8}$$

The left-hand side of (3.7) is clearly independent of the group parameter θ. Thus the right-hand side should also be independent of θ. By performing a Laurent expansion of the right-hand side of (3.7) around $\theta = 0$, we find that setting the coefficients of the θ^{-4} and θ^{-2} terms to zero gives the constraints

$$\int_B c_1^2 = \int_B c_2 \tag{3.9}$$

and

$$\mathrm{Sign}(B) = \int_B c_2 \tag{3.10}$$

respectively, where a sum over each connected component of B is understood in these formulae. The constant term in the Laurent expansion then yields the G-signature theorem:

$$\mathrm{Sign}(X) = \mathrm{Sign}(B)\,. \tag{3.11}$$

In terms of Pontryagin forms, this reads

$$\frac{1}{45}\int_X \left(-p_1^2(X) + 7p_2(X)\right) = \frac{1}{3}\int_B p_1(TB)\,. \tag{3.12}$$

Unfortunately, this is not the correct linear combination of Pontryagin numbers required to prove the relation between the anomalous terms (2.9). We shall therefore need another G-index theorem. It is straightforward to check that

$$\frac{1}{192}\int_X \left(p_1^2(X) - 4p_2(X)\right) = 2\cdot\mathrm{Index}(\mathrm{Dirac}(X)) - \frac{1}{8}\mathrm{Sign}(X) \tag{3.13}$$

in general. Thus we need to study the G-index theorem for the Dirac operator on X.

3.2 The G-Dirac theorem

We take the Dirac operator on the spin manifold X. The details of the G-index theorem in this case were not worked out explicitly in [11]. Rather than present the details (which involve fairly standard calculations), I shall simply state the result. With appropriate conventions, the G-index theorem reduces to

$$\mathrm{Index}(\mathrm{Dirac}(X)) = -\frac{1}{4}\mathrm{cosec}^2\left(\frac{\theta}{2}\right)\int_B \hat{A}(TB)\wedge\mathcal{C}^\theta\,. \tag{3.14}$$

Here $\hat{A}$ is the Dirac genus, given by

$$\hat{A} = 1 - \frac{p_1}{24} + \frac{7p_1^2 - 4p_2}{5760} + \cdots \tag{3.15}$$

and the stable characteristic class $\mathcal{C}^\theta = \mathcal{C}^\theta(c_k)$ is again a polynomial in the Chern classes of V. Specifically, we find that

$$\mathcal{C}^\theta = 1 + \frac{1}{2} i \cot\left(\frac{\theta}{2}\right) c_1 + \frac{1}{4}\mathrm{cosec}^2\left(\frac{\theta}{2}\right)(c_2 - c_1^2) + \frac{1}{8}c_1^2 + \cdots . \tag{3.16}$$

Thus the G-Dirac theorem reads

$$\begin{aligned} \mathrm{Index}(\mathrm{Dirac}(X)) &= \frac{1}{96}\mathrm{cosec}^2\left(\frac{\theta}{2}\right)\int_B p_1(TB) - \frac{1}{32}\mathrm{cosec}^2\left(\frac{\theta}{2}\right)\int_B c_1^2 - \\ &\quad - \frac{1}{16}\mathrm{cosec}^4\left(\frac{\theta}{2}\right)\int_B (c_2 - c_1^2) . \end{aligned} \tag{3.17}$$

Again, we may perform a Laurent expansion of the right-hand side of (3.17) around $\theta = 0$. Setting the coefficients of the θ^{-4} and θ^{-2} terms to zero gives the same constraints (3.9), (3.10) as the G-signature theorem, with the final result that the index of the Dirac operator is zero:

$$\mathrm{Index}(\mathrm{Dirac}(X)) = 0 . \tag{3.18}$$

Putting everything together, we get

$$\frac{1}{192}\int_X \left(p_1^2(X) - 4p_2(X)\right) = -\frac{1}{8}\mathrm{Sign}(X) = -\frac{1}{8}\mathrm{Sign}(B) = -\frac{1}{24}\int_B p_1(TB) . \tag{3.19}$$

Thus the result (2.9) follows if we can show that

$$\int_B p_1(TB) = -\int_B p_1(NB) . \tag{3.20}$$

Again, a sum over each connected component of B is understood in this formula. Thus it remains to prove the last equation, or, equivalently, that

$$\int_B p_1(TY) = 0 . \tag{3.21}$$

Equivalence follows since $TY \mid_B = TB \oplus NB$ and therefore $p_1(TY) \mid_B = p_1(TB) + p_1(NB)$. Assuming for the moment the result (3.21), together with anomaly cancellation in M-theory and string theory, we find that the C-field tadpole is given by

$$\frac{1}{192}\int_X (p_1^2 - 4p_2) + \frac{1}{2}\int_X \left(\frac{G}{2\pi}\right)^2 = \int_B \mathrm{ch}\mathcal{F} \wedge \hat{A}(TB) = \mathrm{Index}(\mathrm{Dirac}(B)) \tag{3.22}$$

the index of the natural Dirac operator on B which couples to the U(1) gauge field.[6] This is clearly an integer. One may therefore cancel the tadpole for C by including $|\mathrm{Index}(\mathrm{Dirac}(B))|$ space-filling M2/D2 branes or anti-branes (depending on the sign of the index) in the vacuum.

There are at least two ways of proving (3.21), which essentially amounts to a statement of charge conservation. The first is an "M-theory" proof, where we use the constraints (3.9), (3.10) of the G-index theorems. The second is a "string theory" proof, which uses only (co)-bordism theory.

[6] More accurately, the "U(1) gauge field" is actually a spin^c connection.

M-theory proof. The first step is to write the Pontryagin class $p_1(NB)$ in string theory in terms of Chern classes in M-theory. That is, we wish to relate a characteristic class of the normal bundle of B in Y to characteristic classes of the normal bundle of B in X, which recall we denoted by V.

We may do this as follows. The structure group of the rank two complex vector bundle V is $\mathrm{U}(2)$. Choose the standard maximal torus $T = (\mathbf{S}^1)^2$ for $\mathrm{U}(2)$ with basic characters x_1, x_2. Then reducing from M-theory to type-IIA corresponds to taking the projectivisation of this structure group. Thus the structure group of the normal bundle of B in Y is $\mathrm{SO}(3) \cong \mathrm{U}(2)/\mathrm{U}(1)$ where the $\mathrm{U}(1)$ denotes the central $\mathrm{U}(1)$ subgroup. The determinant line bundle of V has first Chern class $c_1(V) = x_1 + x_2 \in H^2(B;\mathbb{Z})$. It follows that the basic character for the maximal torus $T = \mathbf{S}^1$ for $\mathrm{SO}(3)$ may be taken to be the anti-diagonal combination $x_1 - x_2$.

Recall now that the first Pontryagin class of a real vector bundle E may be defined in terms of the basic characters x_i of the standard maximal torus of $\mathrm{SO}(n)$ as

$$p_1(E) = x_1^2 + \cdots + x_m^2 \tag{3.23}$$

where $m = \lfloor \frac{n}{2} \rfloor$. Thus

$$p_1(V_{\mathbb{R}}) = x_1^2 + x_2^2 \tag{3.24}$$

and

$$p_1(NB) = (x_1 - x_2)^2 \,. \tag{3.25}$$

It follows that

$$p_1(NB) = 2p_1(V_{\mathbb{R}}) - c_1^2(V) \,. \tag{3.26}$$

Now, we also have

$$p_1(V_{\mathbb{R}}) = -c_2(V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}) = -c_2(V \oplus \bar{V}) = -2c_2(V) + c_1^2(V) \tag{3.27}$$

and so

$$p_1(NB) = 2(-2c_2(V) + c_1^2(V)) - c_1^2(V) = -4c_2 + c_1^2 \,. \tag{3.28}$$

Finally, we may use the constraints (3.9), (3.10) from the G-index theorems to relate c_2, c_1^2 and $\mathrm{Sign}(B)$. We obtain

$$\int_B p_1(NB) = \int_B -4c_2 + c_1^2 = -3\int_B c_2 = -3 \cdot \mathrm{Sign}(B) = -\int_B p_1(TB) \tag{3.29}$$

thus proving (3.21).

String theory proof and bordism theory. We have now completed our task. However, the proof just given relied on the mathematical construction relating string theory with M-theory, together with G-index theory. There is actually a very nice "elementary" interpretation of (3.21) in terms of bordism theory. Specifically, (3.21) corresponds to the fact that a certain bordism invariant must vanish. We finish with a brief explanation.

Recall that the spin six-manifold S bounds a small tubular neighbourhood around the D6-brane B,[7] so that S is a two-sphere bundle over B. Over any fibre of this bundle, we have

$$\int_{\mathbf{S}^2} \frac{G_2}{2\pi} = 1 \tag{3.30}$$

which states that there is a single D6-brane wrapped on B. Now, the RR two-form $a \equiv [G_2/2\pi]$ may be viewed as an element of $H^2(S;\mathbb{Z})$ that extends over $Y \setminus B$. This statement simply says that the only D6-branes present are those wrapped on B (which is implicit in our notation). It follows that the pair (S,a) is zero as an element of the spin bordism group $\Omega_6^{\mathrm{spin}}(K(\mathbb{Z},2))$. This group is non-zero, and to say that $(S,a) = 0 \in \Omega_6^{\mathrm{spin}}(K(\mathbb{Z},2))$ leads to the constraint (3.21).

To explain this in a little more detail, recall that the spin bordism group $\Omega_n^{\mathrm{spin}}(Z)$ of the space Z may be defined by taking pairs (M_n, f), where $f : M_n \to Z$ is a continuous map from the spin n-manifold M_n into Z, subject to the equivalence relation that the pair (M_n, f) is corbordant to zero if and only if there is a spin $(n+1)$-manifold B_{n+1} that bounds M_n and an extension $\tilde{f} : B_{n+1} \to Z$ of the map f. The group structure is simply given by the disjoint union of manifolds.

In the case at hand, we have a spin six-manifold S together with a dimension two cohomology class a on S. The class a may be defined by a map $f : S \to K(\mathbb{Z},2)$ from S to the Eilenberg-MacLane space $K(\mathbb{Z},2)$ that classifies integral two-dimenional cohomology. Equivalently, such classes a are in one-to-one correspondence with complex line bundles over S. The classifying space is $\mathbb{C}P^\infty = K(\mathbb{Z},2)$. Thus, equivalently,

$$(S,a) = 0 \in \Omega_6^{\mathrm{spin}}(\mathbb{C}P^\infty)\,. \tag{3.31}$$

One can show that $\Omega_6^{\mathrm{spin}}(\mathbb{C}P^\infty) = \Omega_8^{\mathrm{spin}^c} \equiv \Omega_8^{\mathrm{spin}^c}(\{\mathrm{point}\})$ is a non-trivial group [12], and therefore that (3.31) is a non-trivial statement.

A key notion in (co)-bordism theory is that of a characteristic number. For example, for the map $f : M \to Z$ one has generalised Pontryagin numbers

$$\int_M f^*(z) \cdot p_\omega(TM) \tag{3.32}$$

where $p_\omega(TM)$ denotes a monomial[8] in Pontryagin numbers of M, and z is any element of $H^*(Z;\mathbb{Z})$. The cohomology ring $H^*(\mathbb{C}P^\infty;\mathbb{Z})$ of the classifying space $\mathbb{C}P^\infty$ is the integral polynomial ring on the generator $x = c_1$. Pulling this back to M via the map f therefore gives the first Chern class a. For us, this is the first Chern class of the M-theory circle bundle. In particular, we have a homomorphism $h : \Omega_6^{\mathrm{spin}}(\mathbb{C}P^\infty) \to \mathbb{Z}$ given by

$$h((S,a)) = \int_S a \wedge p_1(TS)\,. \tag{3.33}$$

[7] We suppress the three-dimensional spacetime M_3 in the argument.

[8] The subscript ω denotes the particular monomial.

If (S, a) is zero as a spin bordism class, the characteristic number $h((S, a))$ must vanish. Using (3.30) we therefore have that

$$h((S,a)) = \int_S a \wedge p_1(TS) = \int_B p_1(TS) \tag{3.34}$$

should vanish, and therefore (3.21) should vanish. Implicit in our calculation here is the fact that $p_1(TS)$ is a pull-back from B (it is also the restriction of $p_1(TY)$ to S).

Notice that h is not the only bordism invariant of (S, a) - it just happens to be the invariant of interest. For example, the characteristic number

$$\int_S a^3 \tag{3.35}$$

should also vanish in the current situation. One can prove that this is indeed the case using the results of section 3.1. The details are left as an exercise for the interested reader.

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References

[1] M.J. Duff, J.T. Liu and R. Minasian, *Eleven-dimensional origin of string/string duality: a one-loop test*, *Nucl. Phys.* **B 452** (1995) 261 [hep-th/9506126].

[2] E. Witten, *Five-brane effective action in M-theory*, *J. Geom. Phys.* **22** (1997) 103 [hep-th/9610234].

[3] C. Vafa and E. Witten, *A one loop test of string duality*, *Nucl. Phys.* **B 447** (1995) 261 [hep-th/9505053].

[4] M.B. Green, J.A. Harvey and G.W. Moore, *I-brane inflow and anomalous couplings on D-branes*, *Class. and Quant. Grav.* **14** (1997) 47 [hep-th/9605033];
R. Minasian and G.W. Moore, *k-theory and Ramond-Ramond charge*, *J. High Energy Phys.* **11** (1997) 002 [hep-th/9710230];
J.F. Morales, C.A. Scrucca and M. Serone, *Anomalous couplings for D-branes and o-planes*, *Nucl. Phys.* **B 552** (1999) 291 [hep-th/9812071];
B. Stefański Jr., *Gravitational couplings of D-branes and o-planes*, *Nucl. Phys.* **B 548** (1999) 275 [hep-th/9812088].

[5] M. Bershadsky, C. Vafa and V. Sadov, *D-branes and topological field theories*, *Nucl. Phys.* **B 463** (1996) 420 [hep-th/9511222].

[6] S.P. de Alwis, *A note on brane tension and M-theory*, *Phys. Lett.* **B 388** (1996) 291 [hep-th/9607011].

[7] S. Gukov and J. Sparks, *M-theory on* Spin(7) *manifolds. I*, *Nucl. Phys.* **B 625** (2002) 3 [hep-th/0109025].

[8] E. Witten, *On flux quantization in M-theory and the effective action*, *J. Geom. Phys.* **22** (1997) 1 [hep-th/9609122].

[9] J.F. Sparks, *M-Theory on* Spin(7) *manifolds*, PhD Thesis, University of Cambridge, Cambridge 2002.

[10] C.A. Scrucca and M. Serone, *Anomalies and inflow on D-branes and o-planes*, *Nucl. Phys.* **B 556** (1999) 197 [hep-th/9903145].

[11] M.F. Atiyah and I.M. Singer, *The index of elliptic operators. 3*, *Annals Math.* **87** (1968) 546–604.

[12] R.E. Stong, *Notes on cobordism theory*, Princeton University Press, Princeton 1968.